Alternative Kinematic Assumptions for the Motions of Structures with Multiple Dynamic Degrees of Freedom Subjected to Seismic Actions

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Abstract: - The paper represents a work of synthesis and research for the earthquake response of structures under various assumptions. The cases of a structure with multiple degrees of freedom, having a rigid body motion between the ground-structure interface and a random, spatial motion are presented. The earthquake induces a spatial, random motion to the ground-structure interface contact points and alternative ways and hypotheses of determining the equations of motion and their solutions are shown. New approaches are needed because the simplified manner in which this subject is usually treated (one degree of freedom or plane structures) is not compatible with the real, physical, phenomenon and may introduce coarse errors in analyses or design. When the non-synchronous nature of motion of the ground-structure interface is important, instead of dealing with the unknowns of relative displacements, absolute displacements should be considered.

Keywords: - relative and absolute displacements, equations of motion, matrices, random, spatial motion

1. INTRODUCTION

The objectives of the paper may be summarized as comparative approaches to the cases of the rigid motion of the interface and the general, non-synchronous motion, respectively, and alternative formats to represent the conventional seismic action and its effects as proposed for the development of codes.

The special feature of the earthquake problem compared to any other form of dynamic loading is that the excitation is applied in the form of support motions rather than by external loads. Earthquake ground motions are usually expressed in terms of three components of translational accelerations. The response of a linear system to these components of input can be computed with the help of the superposition principle, thus adding or superposing the responses calculated for each component. In this manner, the standard analytical problem is reduced to the evaluation of the structural response to a single component of support translation. In the general case of the motion, besides the three translations of the ground-structure interface, three rotations will be added, as the waves propagate through the foundation soils. Unfortunately, no measurements have been made for the magnitude and character of the ground-rotation components and consequently this effect has been accounted for only by tentative order-of-magnitude analyses in which rotational motions were hypothesized from the translational components.

Another assumption related to earthquakes is that the same motion acts simultaneously in all the points of the foundation of the structure. If rotational motions are neglected, this assumption is similar with considering the foundation soil or rock to be rigid, hypothesis acceptable under certain circumstances only (base dimensions of the structure are small relative to the vibration wavelengths in the basement rock). Therefore it is important to develop analysis procedures capable of dealing with multiple support excitations, meaning, with different earthquake inputs applied at separate points of support.

One final factor that should be considered when defining the effective forces developed in a structure by an earthquake is that the ground motions at the base of the structure may be influenced by the motions of the structure itself. Thus the motion introduced at the base of the structure may be different from the free-field motions that would have been observed without the structure. If the foundation rock is firm and the structure is relatively flexible, then the soil-structure interaction effect may be neglected being of slight importance (little energy transmitted into the soil). In general, both the ground motion modification and the interaction effects of a soft layer can be important and must be accounted for in an earthquake response analysis.

Oscillatory motions are generated by two basic categories of actions:

a) Systems of given dynamic forces (applied to some points to which dynamic degrees of freedom - DoF- are associated)

b) Systems of given accelerations applied to the support points.

The disturbing factor is, in this case, the ground motion which is distinguished by a chaotic, transient, spatial character and in the case of high intensities, very violent. This leads to high stresses which in numerous cases imply post-elastic deformations of the structures and thus a nonlinear behavior.

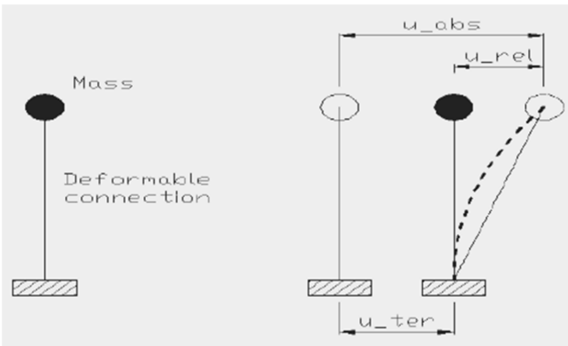

Figure 1. System with SDoF

2. SEISMIC OSCILLATIONS OF MDOF STRUCTRUES

The kinematics of a system with SDof on which a seismic force is applied is illustrated in Figure 1, in which the following displacements appear:

- The displacement of the support point or of the ground, u_{ter}, which plays a role of transport displacement;
- The relative displacement of the mass, u_{rel};
- The absolute displacement of the mass, u_{abs};

$$u_{abs}(t) = u_{ter}(t) + u_{rel}(t) \tag{1}$$

In the equation of motion of the system (1), the inertia forces corresponding to the absolute acceleration and the reactive force corresponding to the relative displacement and relative velocity will occur. The exact expression of the reactive force will correspond to the type of rheological law of the connecting element.

In practice the response may actually be calculated through a step-by-step integration or by a frequency-domain analysis.

The formulation of the earthquake-response analysis of a multiple degrees of freedom system (MDoF) can be carried out in matrix notation in a totally similar manner with the development of the equations for the single DoF system.

When a structure is supported at more than one point and has different ground motions applied at each, the total response of the structure can be obtained by superposition of the response due to each independent support input.

Two types of degrees or freedom (DoF) are considered in this approach [1]:

- DoF which characterize the position of the structure in the hypothesis in which the ground-structure interface is blocked, denoted by $k,l,\ldots=1\ldots n$, to which the vectors $u(t)$ correspond;
- DoF which characterize the position of the ground-structure interface, denoted by $c,d,\ldots=1\ldots m$, to which the vectors $u_{ter}(t)$ correspond.

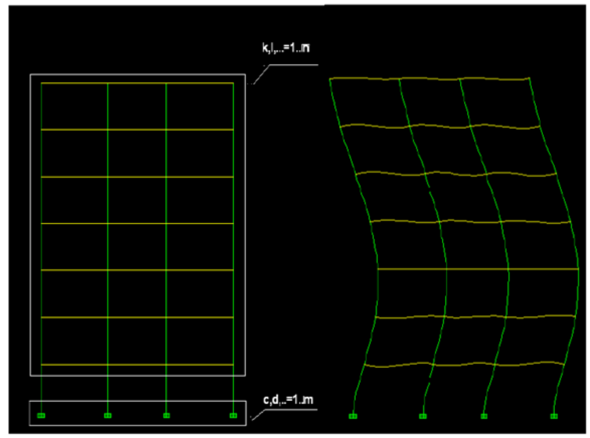

Figure 2. MDoF System and its Deformed Shape

When establishing the equations of motion an artifice is employed: the absolute displacement vector $u(t)$, is decomposed into two terms:

$$u_{abs}(t) = u_{ref}(t) + u_{rel}(t) \tag{2}$$

in which: $u_{ref}(t)$ represents a vector corresponding to a system of displacements compatible with the displacements $u_{ter}(t)$ of the ground-structure interface (in principle an arbitrary vector) and $u_{rel}(t)$ represents a vector corresponding to a system of supplementary displacements.

The matrix equation of motion, with respect to the DoF $k,l,\ldots=1\ldots n$ is:

$$\begin{aligned} M\left(\ddot{u}_{ref} + \ddot{u}_{rel}\right) + C\left(\dot{u}_{ref} + \dot{u}_{rel}\right) + \\ + K\left(u_{ref} + u_{rel}\right) = 0 \end{aligned} \tag{3}$$

$$\hat{Z}(p)\left[\hat{u}_{ref}(p) + \hat{u}_{rel}(p)\right] = 0 \tag{4}$$

$$\hat{Z}(p) = p^2 M + \hat{K}(p) \tag{5}$$

where $\hat{Z}(p)$ - represents the impendance matrix, using Laplace-Carson images for ease of computation [1,2].

If in the left member only the unknown vector $u_{rel}(t)$ is retained, the vector $u_{ref}(t)$ being given, then one obtains:

$$M\ddot{u}_{rel} + C\dot{u}_{rel} + Ku_{rel} = f_{ref}(t) \tag{6}$$

$$f_{ref}(t) = -M\ddot{u}_{ref} - C\dot{u}_{ref} - Ku_{ref} \quad (7)$$

For the Laplace-Carson images the equations become:

$$\hat{Z}(p)\hat{u}_{rel}(p) = \hat{f}_{ref}(p) \quad (8)$$

$$\hat{f}_{ref}(p) = -\hat{Z}(p)\hat{u}_{ref}(p) \quad (9)$$

The process is general. The arbitrary choosing of u_{ref} (t) may be disadvantageous when it comes to actual computations and thus particularizations which lead to simplifications of the computations are required.

2.1. Rigid Solid Motion

The first simplification [1]: the motion of the ground-structure interface is a motion of a rigid solid. It is advantageous if u_{ref} (t) represents a motion of rigid solid as well, and, thus results a vector u_{rig} (t) (motion compatible with u_{ter} (t)).

$$u_{rig}(t) = Gu_{ter}(t) \quad (10)$$

where **G** represents a rectangular matrix corresponding to a roto-translation of rigid solid.

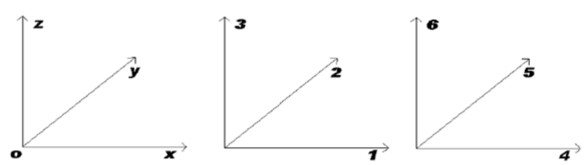

Figure 3. Order of the Axes when Considering the Matrix Components [1]

If u_{ter} (t) has components denoted as in Figure 3, considered with respect to a given Cartesian orthogonal reference system and if u_{rig} (t) has components corresponding to the spatial displacements of some nodes, then G will be a matrix with 6 columns having the column structure made from quadratic sub matrices [1,4] G_{i0}, corresponding to i_0 nodes:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & z_{i0} & -y_{i0} \\ 0 & 1 & 0 & -z_{i0} & 0 & x_{i0} \\ 0 & 0 & 1 & y_{i0} & -x_{i0} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (11)$$

Thus, the relative displacements u_{rel} (t) are given a more simple meaning becoming relative displacement with respect to a rigid roto-translation compatible with the rigid roto-translation of the ground, u_{ter} (t).

The equation of motion thus becomes:

$$M\ddot{u}_{rel} + C\dot{u}_{rel} + Ku_{rel} = f_{rig}(t) \quad (12)$$

$$f_{rig}(t) = -M\ddot{u}_{rig}(t) = -MG\ddot{u}_{ter}(t) \quad (13)$$

The relations may be further simplified through different assumptions in function of the technical interest of the problem [5]. The ground motion is considered: a plane (2D) motion of rigid solid (in a vertical dynamic symmetry plane) or as a unidirectional motion of translation.

2.2. Spatial Motion

u_{ref} (t) is identically null (the nodes are fixed or blocked) and thus u_{rel} (t) will coincide with the absolute vector $u_{abs}(t)$.
The equation becomes:

$$M\ddot{u}_{abs} + C\dot{u}_{abs} + Ku_{abs} = f_{fix}(t) \quad (14)$$

$$\hat{Z}(p)\hat{u}_{abs}(p) = \hat{f}_{fix}(p) \quad (15)$$

$$\hat{f}_{fix}(p) = \hat{R}(p)\hat{u}_{ter}(p) \quad (16)$$

The matrix $\hat{R}(p)$ [1] is determined considering that:
a) In its determination the property of symmetry may be used;
b) A column of index c of the matrix $\hat{R}(p)$ represents the image of the reactions of the structure along the DoF $k,l,...(=1...n)$ when the displacements along these DoFs are blocked and on the direction of the degree of freedom c of the interface a displacement is applied whose image is 1 (meaning a displacement varying after Heaviside's step function).
c) A line of index k of the matrix $\hat{R}(p)$ represents the image of the forces applied to the ground along the DoF $c,d,...(=1...m)$, when the displacements of the interface are blocked and when along the degree of freedom k of the structure a displacement whose image is 1(Heaviside) is applied.
d) In the case of an elastic structure the matrix becomes real and constant, R, and in the case of a visco-elastic structure (Kelvin model), the matrix will become $\hat{R}(p) = R + pS$ in which the matrix S corresponds to the viscous stiffness of the structure.

3. SOLVING THE EQUATIONS OF MOTION

Any structure of arbitrary form can be treated as a SDoF system if it is assumed that its displacements are restricted to a single natural shape of vibration. This generalized-coordinate approach can be used effectively in earthquake engineering considering the only special problem, in this case, the evaluation of the generalized force resulting from the support excitation. The basic assumption of the approximation is that the displacements are given by the product of a single natural shape of vibration function and generalized coordinate amplitude. The method of solving the general equations of motion is that of employing the solutions of the eigenvalue problems. Any vector may be represented under the form [1,5]:

$$u(t) = \sum_r^{1,n} v_r{}^{(M)} q_r{}^{(M)}(t) \qquad (17)$$

where the functions $q_r(t)$ and $q_r{}^{(M)}(t)$ are called normal (modal) and normalized coordinates; v_r are the eigenvectors and, respectively, $v_r{}^{(M)}$ the normalized eigenvectors with respect to the mass matrix, M.

Similarly, the vector for the Laplace-Carson images can be written as:

$$\hat{u}(p) = \sum_r^{1,n} v_r{}^{(M)} \hat{q}_r{}^{(M)}(p) \qquad (18)$$

Having written the general form for the vector as well as for its image, one can adopt similar relations for the solutions of the vectors $u_{rel}(t)$ and $u_{abs}(t)$.

The general equation becomes [1,2,3]:

$$M\ddot{u} + C\dot{u} + ku = f(t) \rightarrow \ddot{q}_r{}^{(M)} + 2n_r\omega_r\dot{q}_r{}^{(M)} + \omega_r^2 q_r{}^{(M)} = g_r{}^{(M)}(t) \qquad (19)$$

in which: n_r represents the critical damping ratio and ω_r the natural frequency.

And the free term is:

$$g_r{}^{(M)}(t) = v_r{}^{(M)T} f(t) \qquad (20)$$

For the two previously discussed cases, the free terms are:

$$g_{rel,r}{}^{(M)}(t) = v_r{}^{(M)T} f_{rig} = -v_r{}^{(M)T} MG\ddot{u}_{ter}(t) \qquad (21)$$

$$g_{abs,r}{}^{(M)}(t) = v_r{}^{(M)T} f_{fix}(t) = -v_r{}^{(M)T}[Ru_{ter}(t) + S\dot{u}_{ter}(t)] \qquad (22)$$

In order to reduce the problem to a number of n differential equations with one unknown the solutions found for the systems with one degree of freedom will be used [1,2].

$$\ddot{u}_{rel}(t) = -\sum_r^{1,n} v_r{}^{(M)} v_r{}^{(M)T} MG \left\{ \ddot{u}_{ter}(t) + \omega_r' \int_0^t e^{-n_r\omega_r(t-t')} \left[\frac{1-2n_r^2}{\sqrt{1-n_r^2}} \sin\omega_r'(t-t') + 2n_r\cos\omega_r'(t-t') \right] \ddot{u}_{ter}(t')dt' \right\} \qquad (23)$$

In the hypothesis of an elastic structure:

$$\ddot{u}_{abs}(t) = \sum_r^{1,n} v_r{}^{(M)} v_r{}^{(M)T} R \left\{ \ddot{u}_{ter}(t) - \omega_r' \int_0^t e^{-n_r\omega_r(t-t')} \left[\frac{1-2n_r^2}{\sqrt{1-n_r^2}} \sin\omega_r'(t-t') + 2n_r\cos\omega_r'(t-t') \right] \ddot{u}_{ter}(t')dt' \right\} \qquad (24)$$

Finally, for the Laplace-Carson images the solutions are:

$$\hat{u}_{rel}(p) = \hat{Z}^{-1}(p)\hat{f}_{rig}(p) = -\sum_r^{1,n} \frac{v_r{}^{(M)} v_r{}^{(M)T}}{p^2+\hat{\lambda}(p)} MGp^2\hat{u}_{ter}(p) \qquad (25)$$

$$\hat{u}_{abs}(p) = \hat{Z}^{-1}(p)\hat{f}_{fix}(p) = -\sum_r^{1,n} \frac{v_r{}^{(M)} v_r{}^{(M)T}}{p^2+\hat{\lambda}(p)} R(p)\hat{u}_{ter}(p) \qquad (26)$$

$$\hat{\lambda}_r(p) = v_r{}^{(M)T}\hat{K}(p)v_r{}^{(M)} \qquad (27)$$

4. EXCITATION BY RIGID BASE ROTATION

To deal with any form of earthquake input other than rigid-base translation, the usual discussed case, it is necessary only to define the matrix r appropriate to the new support-motion condition. The total displacement vector is expressed as a pseudo-static displacement plus the dynamic motions, and consequently the pseudo-static influence coefficients, r_i, contain input on the nature of the earthquake.

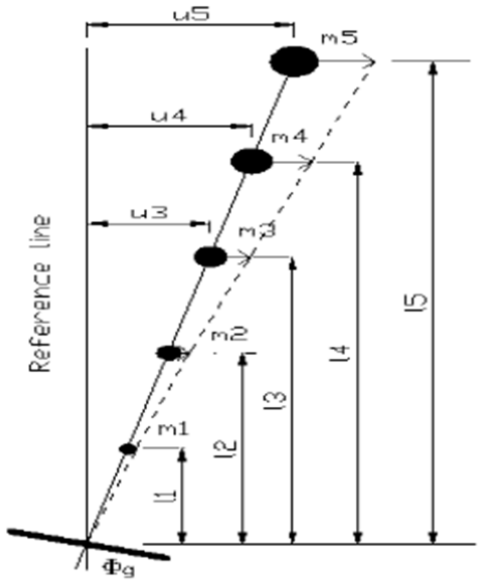

Figure 4. Lumped Mass System Subjected to Base Rotation

Now, considering the case from Figure 4, a cantilever like structure subjected to small amplitude base rotation, the pseudo-static displacements in this case are given by a rigid rotation about the base [2]:

$$u^s = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \end{bmatrix} \theta_g \quad (28)$$

Thus the influence coefficient matrix is merely the listing of the height of the masses above the base:

$$r^T = [l_1 \quad l_2 \quad l_3 \quad l_4 \quad l_5] \quad (29)$$

After this vector is determined, the analysis steps and procedures remain the same as for the systems with a rigid-base translation. The only important factor that needs to be taken into consideration is that the effective earthquake moments are induced in proportion to the rotational inertia of the masses (J) as well as the effective translational forces which result from the translational inertia.

5. APPLICATIONS

5.1. Translation

The structure in Figure 5 is considered in the hypothesis of inextensible columns and of perfectly rigid horizontal beams which concentrate the masses. It is admitted that the motion of the ground is produced only on the horizontal translation direction. The structure has 3DoFs, corresponding to the displacements u_1, u_2, and u_3.

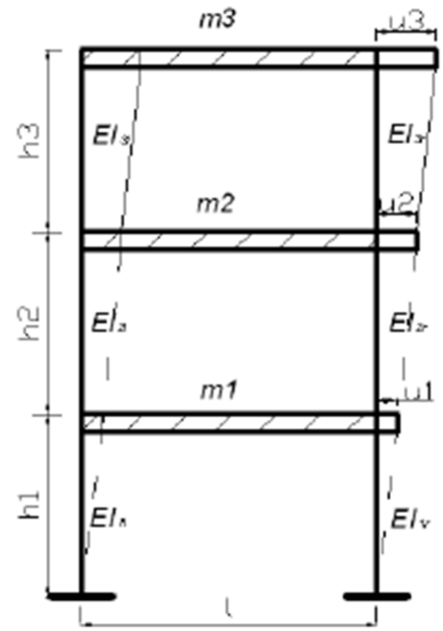

Figure 5. Structural System for Applications

The G matrix will be replaced by the vector g and thus the free term from equation (12) will become:

$$g = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (30)$$

$$f_{rig}(t) = -MG\ddot{u}_{ter}(t) = -M \begin{bmatrix} \ddot{u}_{ter} \\ \ddot{u}_{ter} \\ \ddot{u}_{ter} \end{bmatrix} \quad (31)$$

The mass matrix is diagonal and thus:

$$f_{rig}(t) = -\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{u}_{ter} \\ \ddot{u}_{ter} \\ \ddot{u}_{ter} \end{bmatrix} = -\begin{bmatrix} m_1\ddot{u}_{ter} \\ m_2\ddot{u}_{ter} \\ m_3\ddot{u}_{ter} \end{bmatrix} \quad (32)$$

5.2. Rigid Solid Motion

The same structure is considered but subjected to a seismic motion represented by a plan movement, of rigid solid, of the interface. The vector $\ddot{u}_{ter}(t)$ will have 3 components which correspond to the degrees of freedom 1, 3 and 5 of Figure 3. The fact that the columns are inextensible leads to a decrease in the number of degrees of freedom (no vertical displacement of the structure). The structure behaves similarly when considering the rotations in the vertical plane. The structure follows the movement of the ground. Thus, if the 3 components of the acceleration are denoted by $\ddot{u}_{ter}(t)$, $\ddot{w}_{ter}(t)$ and $\ddot{\varphi}_{ter}(t)$, the matrix G will have 3 lines corresponding to the DoFs of the structure and two columns corresponding to the two acceleration components.

$$G = \begin{bmatrix} 1 & h_1 \\ 1 & h_1 + h_2 \\ 1 & h_1 + h_2 + h_3 \end{bmatrix} \tag{33}$$

5.3. Space Oscillations

The same structure as before is considered and in this case oscillates in space. Thus, the beams shall have each 3 additional DoFs corresponding to the directions 2, 4, and 6 from Figure 3. The components of the motion of a rigid beam will be considered in the order 1, 2, 4 and 6. The matrix G will have 3 x 4 = 12 lines. From the 6 degrees of the interface motion the 3rd is neglected as in the previous case (inextensible columns), thus leaving only 5 DoFs corresponding to the directions 1, 2, 4, 5 and 6 for which the following notations are used: $\ddot{u}_{ter}(t)$, $\ddot{v}_{ter}(t)$, -$\ddot{\alpha}_{ter}(t)$, $\ddot{\varphi}_{ter}(t)$, and $\ddot{\beta}_{ter}(t)$. The matrix G will have the following expression:

$$G = \begin{bmatrix} 1 & 0 & 0 & h_1 & 0 \\ 0 & 1 & -h_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & h_1 + h_2 & 0 \\ 0 & 1 & -(h_1 + h_2) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & h_1 + h_2 + h_3 & 0 \\ 0 & 1 & -(h_1 + h_2 + h_3) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{34}$$

5.4. Arbitrary Plan Motion

The last case considered is the general one having the same structure, this time subjected to an arbitrary plan motion of the interface. Not having the rigid body displacement condition, each column will have 3 DoFs on the directions 1, 3 and 5 from Figure 3, and if for the columns in the left part and in the right part additional indices are used (l and r) the vector $\ddot{u}_{ter}(t)$ will have, in order, the following components $\ddot{u}_{ter,l}(t)$, $\ddot{w}_{ter,l}(t)$, $\ddot{\varphi}_{ter,l}(t)$, $\ddot{u}_{ter,r}(t)$, $\ddot{w}_{ter,r}(t)$, $\ddot{\varphi}_{ter,r}(t)$. Because the columns are inextensible the beams can only deform following 1 DoF (horizontal translation). If the structure is ideally elastic, the matrix R will be a real and constant one. Its components will be determined according to the 3rd observation when describing the matrix theoretically. A unit horizontal translation will lead to horizontal forces and couples applied to the ground, determined by the bending of the first floor columns. On the other hand there is a vertical couple transmitted to the ground, which, with the local couples equates the resultant couple of the external forces applied to the structure in order to deform with a unit translation which determines the components of the matrix R.

$$R = \begin{bmatrix} \frac{12EI_{1l}}{h_1^3} & \frac{6(EI_{1l}+EI_{1r})}{h_1^2 l} - \frac{6(EI_{2l}+EI_{2r})}{h_2^2 l} - \frac{6(EI_{3l}+EI_{3r})}{h_3^2 l} & \frac{6EI_{1l}}{h_1^2} & \frac{12EI_{1r}}{h_1^3} \\ 0 & \frac{6(EI_{2l}+EI_{2r})}{h_2^2 l} - \frac{6(EI_{3l}+EI_{3r})}{h_3^2 l} & 0 & 0 \\ 0 & \frac{6(EI_{3l}+EI_{3r})}{h_3^2 l} & 0 & 0 \end{bmatrix} \tag{35}$$

6. CONCLUSIONS

The paper represents a work of synthesis and research for the earthquake response of structures under various assumptions. The simple case of a structure with a SDoF was presented and then an extension was made to the multiple degrees of freedom, considering a rigid body motion between the ground-structure interface and then a random, spatial motion. At the end, a solved problem was illustrated for each theoretically discussed case.
As a general conclusion, this paper wished to present alternative and viable solutions to the simplified manner in which the problem of seismic action and ground-structure interface motion is generally presented in design codes.

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