Energy Flux for Guided Waves Determined by the Finite Elements Method

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Abstract: - The present paper presents in an informative manner, an efficient method providing the dispersion curves for a planar multilayered structure in vacuum, made of isotropic homogeneous visco-elastic materials. The energy flux of the resulting generalized Lamb waves is then determined for each frequency in the selected range. The energy flux distribution, as function of the position along the normal direction on the laminar structure, can thus be plotted. The flux integral along the cross-section is obtained and represented as function of frequency for all propagating modes.

Keywords: Energy flux, guided waves, finite elements method

1. INTRODUCTION

All guided waves applications require the information from related dispersion curves, representing either the wavenumbers, or wave phase/group velocities, as functions of frequency [1], [2]. These dispersion curves are obtained by solving the specific dispersion equations, usually by numerical methods, but for multilayered structures this task can prove to be difficult.

The present paper presents an efficient method providing the dispersion curves for a planar multilayered structure in vacuum, made of isotropic homogeneous visco-elastic materials. The energy flux of the resulting generalized Lamb waves is then theoretically and numerically determined for each frequency in the selected range. Using the dispersion curves, the displacements, strains and stress components can be determined at each frequency, for each mode. The energy flux distribution, as function of the position along the normal direction on the laminar structure, is then plotted. The flux integral can then be represented as function of frequency for all propagating modes.

The purpose of the paper is to give an explicit and detailed algorithm, based on the so-called Semi-Analytical Finite Elements Method (SAFEM) which is easily implementable in COMSOL, a commercial available software dedicated to Finite Elements Method (FEM) applications. Other methods or algorithms to compute the dispersion curves were developed [3], [4], [5], etc. The detailed steps used in our case and the given example, are intended for master, Ph.D. students and researchers, remarking that the algorithm and results concerning the energy flux frequency dependency are not mentioned to our knowledge in reference textbooks.

2. LAMB WAVES IN MULTILAYERED LAMINAR STRUCTURES

An isotropic homogeneous plate of thickness H is considered.

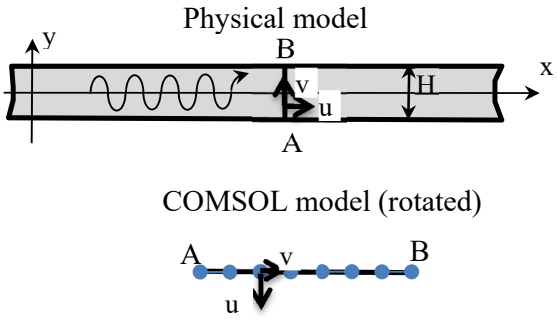

Figure 1. Elastic lamina and Lamb wave propagating along the Ox axis (top). The FEM model (bottom)

The displacements field for the Lamb waves propagating along the Ox direction, can be considered as planar waves:

$$\begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} = A \begin{pmatrix} U(y) \\ V(y) \end{pmatrix} e^{i(kx-\omega t)} \tag{1}$$

in which k is the wavenumber and $\omega = 2\pi f$, in which f is the wave frequency. The functions *U(y)* and *V(y)* represent the longitudinal and transversal component respectively, of the local displacement vector, which has in general a complex valued amplitude A, accounting thus for the wave phase.
The partial derivatives with respect to *x*, which are used for strain computation, are:

$$\frac{\partial}{\partial x}\begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} = ikA \begin{pmatrix} U(y) \\ V(y) \end{pmatrix} e^{i(kx-\omega t)} \tag{2}$$

and their respective time derivatives are:

$$\begin{aligned} &\frac{\partial}{\partial t}\begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} \equiv \begin{pmatrix} \dot{u}(x,y,t) \\ \dot{v}(x,y,t) \end{pmatrix} \\ &= -i\omega \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} = -i\omega A \begin{pmatrix} U(y) \\ V(y) \end{pmatrix} e^{i(kx-\omega t)} \\ &\frac{\partial^2}{\partial t^2}\begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} \equiv \begin{pmatrix} \ddot{u}(x,y,t) \\ \ddot{v}(x,y,t) \end{pmatrix} \\ &= -\omega^2 A \begin{pmatrix} U(y) \\ V(y) \end{pmatrix} e^{i(kx-\omega t)} \end{aligned} \tag{3}$$

The differential equations of motion and stresses in the plane strain hypothesis, are valid also in the particular case of Lamb waves [6]:

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} &= \rho \ddot{u} \\ \frac{\partial \sigma_{21}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} &= \rho \ddot{v} \end{aligned} \tag{4}$$

$$\begin{aligned} \sigma_{11} &= \sigma_{xx} = C_{11}\frac{\partial u}{\partial x} + C_{12}\frac{\partial v}{\partial y} \\ &= A\left(C_{11} ik_x U + C_{12}\frac{\partial V}{\partial y}\right)\exp\left[i\left(k_x x - \omega t\right)\right] \\ &= S_{xx}\exp\left[i\left(k_x x - \omega t\right)\right] \\ \sigma_{12} &= \sigma_{xy} = C_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ &= AC_{66}\left(\frac{\partial U}{\partial y} + ik_x V\right)\exp\left[i\left(k_x x - \omega t\right)\right] \\ &= S_{xy}\exp\left[i\left(k_x x - \omega t\right)\right] \\ \sigma_{22} &= \sigma_{yy} = C_{12}\frac{\partial u}{\partial x} + C_{22}\frac{\partial v}{\partial y} \\ &= A\left(C_{12} ik_x U + C_{11}\frac{\partial V}{\partial y}\right)\exp\left[i\left(k_x x - \omega t\right)\right] \\ &= S_{yy}\exp\left[i\left(k_x x - \omega t\right)\right] \end{aligned} \tag{5}$$

The elasticity coefficients for isotropic materials C_{11}, C_{66}, (C_{12}=C_{11}-2C_{66}) can be computed from experimental data, such as the Young modulus and Poisson coefficient, or from ultrasonic experiments providing c_L and c_T, the longitudinal and shear waves velocities respectively. For visco-elastic materials, these coefficients should be set as complex values, the imaginary part accounting for the dissipation.

Applying the derivation rules in this case, these differential equations along the Ox and Oy axis become:

$$\begin{aligned} &ik\left[C_{11} ikU + C_{12}\frac{\partial V}{\partial y}\right] \\ &+\frac{\partial}{\partial y}\left[C_{66}\left(\frac{\partial U}{\partial y} + ikV\right)\right] = -\rho\omega^2 U \\ &ik\left[C_{66}\left(\frac{\partial U}{\partial y} + ikV\right)\right] \\ &+\frac{\partial}{\partial y}\left[C_{12} ikU + C_{11}\frac{\partial V}{\partial y}\right] = -\rho\omega^2 V \end{aligned} \tag{6}$$

As can be seen, the wave amplitude was cancelled, meaning that the modal displacements and stresses will be determined with the approximation of a multiplicative constant. However, the modal displacements U(y) and V(y) are complex functions of the transverse coordinate, which means that relative phase between them are determined, e.g. U can be only imaginary, whereas V is a real function, in which case the phase lag

between the two components of the displacement field is π/2. The equation (6) can be rewritten in COMSOL formulation as:

$$\begin{aligned}&\frac{\partial}{\partial y}\left[C_{66}\frac{\partial U}{\partial y}\right]+ik\frac{\partial}{\partial y}\left[C_{66}V\right]\\&+ik\left[C_{12}\frac{\partial V}{\partial y}\right]-k^2\left[C_{11}U\right]=-\rho\omega^2U\\&\frac{\partial}{\partial y}\left[C_{11}\frac{\partial V}{\partial y}\right]+ik\frac{\partial}{\partial y}\left[C_{12}U\right]\\&+ik\left[C_{66}\frac{\partial U}{\partial y}\right]-k^2\left[C_{66}V\right]=-\rho\omega^2V\end{aligned}\tag{7}$$

Multilayered laminar structures can be modeled by similar equations, in which only the elastic constants are different but constant over each layer. Perfect bonding condition between layers is represented by usual displacement and strain continuity between layers.

3. DISPERSION CURVES COMPUTED USING A ONE-DIMENSIONAL FEM MODEL

The dispersion curves can be obtained in COMSOL using as FEM model, only a line segment AB perpendicular on the laminar structure, using the eigenvalue solver in the Partial Differential Equations module. It must be noted that in the COMSOL module, the line segment is plotted along the horizontal direction, as shown on Figure 1. This means that the physical transversal coordinate y from Figure 1 is represented by the variable *x* and the partial derivatives $U_{,x}$ and $V_{,x}$ available for post-processing correspond to $U_{,y}$ and $V_{,y}$ in the physical model. Equations (7) can be cast in matrix form, with the mentioned change of variable:

$$\begin{aligned}&\frac{\partial}{\partial x}\left[\begin{bmatrix}C_{66}&0\\0&C_{11}\end{bmatrix}\begin{bmatrix}\frac{\partial U}{\partial x}\\\frac{\partial V}{\partial x}\end{bmatrix}+ik\begin{bmatrix}0&C_{66}\\C_{12}&0\end{bmatrix}\begin{bmatrix}U\\V\end{bmatrix}\right]\\&+ik\begin{bmatrix}0&C_{12}\\C_{66}&0\end{bmatrix}\begin{bmatrix}\frac{\partial U}{\partial x}\\\frac{\partial V}{\partial x}\end{bmatrix}+\begin{bmatrix}\rho\omega^2&0\\0&\rho\omega^2\end{bmatrix}\begin{bmatrix}U\\V\end{bmatrix}\\&=k^2\begin{bmatrix}C_{11}&0\\0&C_{66}\end{bmatrix}\begin{bmatrix}U\\V\end{bmatrix}\end{aligned}\tag{8}$$

The following COMSOL general expression for an eigenvalue problem is given in ref. [7]:

$$\begin{aligned}&\nabla\cdot\left[-[c]\nabla u-[\alpha]u+[\gamma]\right]+[\beta]\cdot\nabla u\\&+[a]u=[d_a]\lambda u-[e_a]\lambda^2u\end{aligned}\tag{9}$$

Changing sign and grouping the useful matrices, the following shorter form represents the problem:

$$\nabla\cdot\left[C\nabla u+ikLu\right]+ikB\cdot\nabla u+Au=k^2[D]u\tag{10}$$

using the following notations:

$$\begin{aligned}&[C]=\begin{bmatrix}C_{66}&0\\0&C_{11}\end{bmatrix};\ [L]=\begin{bmatrix}0&C_{66}\\C_{12}&0\end{bmatrix};\\&[B]=\begin{bmatrix}0&C_{12}\\C_{66}&0\end{bmatrix};\ [A]=\begin{bmatrix}\rho\omega^2&0\\0&\rho\omega^2\end{bmatrix};\\&[D]=\begin{bmatrix}C_{11}&0\\0&C_{66}\end{bmatrix}\end{aligned}\tag{11}$$

The eigenvalue problem will be solved for eigenvalues *k* (the wavenumbers), for a given angular frequency ω. It is clear that matrices *ikLu* or *ikB* have no equivalent in this COMSOL formulation. It is however possible to solve the problem, introducing another set of variables $\begin{bmatrix}U_1&V_1\end{bmatrix}^T$ by the relation:

$$k[M]\begin{bmatrix}U\\V\end{bmatrix}=[M]\begin{bmatrix}U_1\\V_1\end{bmatrix},\tag{12}$$

in which [M] is an arbitrary nonsingular matrix. It can be chosen for example $[M]=[A]$. Thus the COMSOL displacements vector will have four components, requiring more computer RAM memory, but has the advantage of including the matrix expressions *ikLu* and *ikB* in equation (8) at the correct locations corresponding to the involved stresses. The expression *ikLu* from (10), is thus included in the model as:

$$\begin{bmatrix}\begin{bmatrix}0&0\\0&0\end{bmatrix}&\begin{bmatrix}0&0\\0&0\end{bmatrix}\\\begin{bmatrix}0&0\\0&0\end{bmatrix}&[iL]\end{bmatrix}\begin{bmatrix}U\\V\\U_1\\V_1\end{bmatrix},\tag{13}$$

using a four by four coefficient matrix. In a similar manner the imaginary matrix *ikB* is included in the model. Returning to the COMSOL equation general form (9):

$$\nabla\cdot\left[-c\nabla\tilde{U}-\alpha\tilde{U}\right]+\beta\cdot\nabla\tilde{U}+a\tilde{U}=kd_a\tilde{U},\tag{14}$$

which is written for the extended set of modal displacements

$\tilde{U} = \begin{bmatrix} U & V & U_1 & V_1 \end{bmatrix}^T = \begin{bmatrix} U & V & kU & kV \end{bmatrix}^T$ with the new matrices:

$$d_a = \begin{bmatrix} 0 & D \\ A & 0 \end{bmatrix};\ c = \begin{bmatrix} -C & 0 \\ 0 & 0 \end{bmatrix};\ \alpha = \begin{bmatrix} 0 & -iL \\ 0 & 0 \end{bmatrix};$$
$$\beta = \begin{bmatrix} 0 & iB \\ 0 & 0 \end{bmatrix};\ a = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \quad (15)$$

in which 0 represents a 2 by 2 null matrix. The boundary conditions if the two surfaces are free, as in most practical cases, correspond to Neumann boundary conditions:

$$-[c]\nabla\tilde{U} - [\alpha]\tilde{U} = [0]. \quad (16)$$

Solving for the eigenvalues in a parametric sweep over the selected frequency range, will provide the full set of complex wavenumbers, among which there are the real valued ones. The real valued wavenumbers, with positive or negative sign, correspond to propagative modes in positive or negative Ox directions, as shown on Figure 1.

4. THE ENERGY FLUX THROUGH THE CROSS-SECTION

The instantaneous value of the energy flux or power per lamina cross section area of unit depth, of a mode of order q, deduced from the general form given by Auld [2], is:

$$\tilde{P}_q(\omega,t) = \int_{-H/2}^{H/2} \left[\mathrm{Re}\left(\sigma_{xx}^q\right)\mathrm{Re}\left(\dot{u}_q\right) + \mathrm{Re}\left(\sigma_{xy}^q\right)\mathrm{Re}\left(\dot{v}_q\right) \right] dy \quad (17)$$

This formula refers to real valued functions of position and time and assumes here and in the following to be computed per unit depth (1m), in agreement with the plain strain hypothesis.

However, by solving the dispersion equation, even for propagating modes with real $\underline{k}_x$, the displacements functions $U = U^r + iU^i$, $V = V^r + iV^i$ as well as the stress functions $S_{xx}^r + iS_{xx}^i$ and $S_{xy}^r + iS_{xy}^i$ are complex valued functions. Their imaginary parts are offering information on the phase of the respective quantity (displacement, stress), relative to another of the same mode.

For the physical point of view, in formula (17) only products of real parts from the stresses and displacements are required. Since for two arbitrary complex numbers z_1 and z_2, $\overline{z_1 z_2} = \overline{z}_1\overline{z}_2$, in which the overbar stands for complex conjugate, some mathematical transformations are required for a simpler formula. At any given x position along the lamina, the energy flux is the same, if there is no dissipation mechanism involved. For this reason, it can be assumed an arbitrarily selected cross-section (e.g. $x=0$) in the following. The first product from (17) can be written, for an arbitrary mode q, which index is ignored in the following:

$$\begin{aligned} &\mathrm{Re}\left(\sigma_{xx}\right)\mathrm{Re}\left(\dot{u}\right) \\ &= \mathrm{Re}\left[\left(S_{xx}^r + iS_{xx}^i\right)\exp\left(-i\omega t\right)\right] \\ &\cdot \mathrm{Re}\left[-i\omega\left(U^r + iU^i\right)\exp\left(-i\omega t\right)\right] \\ &= \omega\left[S_{xx}^r \cos\omega t + S_{xx}^i \sin\omega t\right] \\ &\cdot\left[U^i \cos\omega t - U^r \sin\omega t\right] \\ &= \omega\left\{\begin{array}{l} S_{xx}^r U^i \cos^2\omega t - S_{xx}^i U^r \sin^2\omega t + \ldots \\ \qquad + \left(S_{xx}^i U^i - S_{xx}^r U^r\right)\cos\omega t \sin\omega t \end{array}\right\} \end{aligned} \quad (18)$$

The superscript r indicates the real part and i the imaginary part of the complex valued function. For a periodic wave, it is practically important to determine the averaged energy flux during one period, as defined in an equivalent form by Achenbach [1], pp.214:

$$\begin{aligned} P_q(\omega) &= -\frac{1}{T}\int_t^{t+T} \tilde{P}_q(\omega,t)\,dt \\ &= -\frac{1}{T}\int_t^{t+T}\left(\int_{-H/2}^{H/2}\left[\mathrm{Re}\,\sigma_{xx}\,\mathrm{Re}\,\dot{u} + \mathrm{Re}\,\sigma_{zx}\,\mathrm{Re}\,\dot{v}\right]dy\right)dt \end{aligned} \quad (19)$$

Replacing in (19) the expression (18) and a similar one for $\mathrm{Re}\,\sigma_{zx}\,\mathrm{Re}\,\dot{u}_z$, then the time average can be obtained using simple formulas, detailed in the Annex (a):

$$P_q(\omega) = -\frac{\omega}{2}\left(\int_{-H/2}^{H/2}\left[S_{xx}^r U^i - S_{xx}^i U^r + S_{xy}^r V^i - S_{xy}^i V^r\right]dy\right) \quad (20)$$

This formula can be cast into several equivalent forms, using the transformations detailed in Annex (b):

$$P_q(\omega) = -\frac{\omega}{2}\mathrm{Im}\left(\int_{-H/2}^{H/2}\left[\overline{S}_{xx}U + \overline{S}_{xy}V\right]_q dy\right), \quad (21)$$

or using the conjugate of displacements:

$$P_q(\omega) = \frac{\omega}{2}\mathrm{Im}\left(\int_{-H/2}^{H/2}\left[S_{xx}\overline{U} + S_{xy}\overline{V}\right]_q dy\right). \quad (22)$$

In some cases, it is useful to use an equivalent combination of these two forms [8]:

$$P_q(\omega) = \\ = \frac{\omega}{4}\mathrm{Im}\left(\int_{-H/2}^{H/2}\left[S_{xx}\bar{U} - \bar{S}_{xx}U + S_{xy}\bar{V} - \bar{S}_{xy}V\right]_q dy\right). \quad (23)$$

5. NUMERICAL EXAMPLE

5.1 Dispersion curves

As numerical example, an aluminum plate with 5mm thickness and 2600 kg/m^3 mass density is selected. The elastic constants are: C11=113.97 GPa, C66=26.91 GPa, C12=60.15 GPa.

The wavenumbers obtained according to formula (14), using a "Parametric sweep" for the frequency range 2-1000 kHz with a step of 2 kHz are shown on Figure 2.

The program provides positive and negative wavenumbers as real solutions. Each complex wavenumber is determined as four numbers: *k'+ik", k'-ik"; -k'+ik"; -k'-ik"*, in which *k'* and *k"* are the real and imaginary parts of the wavenumber respectively.

Only two complex roots correspond to attenuating (evanescent) waves propagating towards the positive or negative direction respectively. For this reason it is necessary to request a minimum of 20 eigenvalues to the COMSOL solver, to obtain the results shown on Figure 2. The solver is sorting the

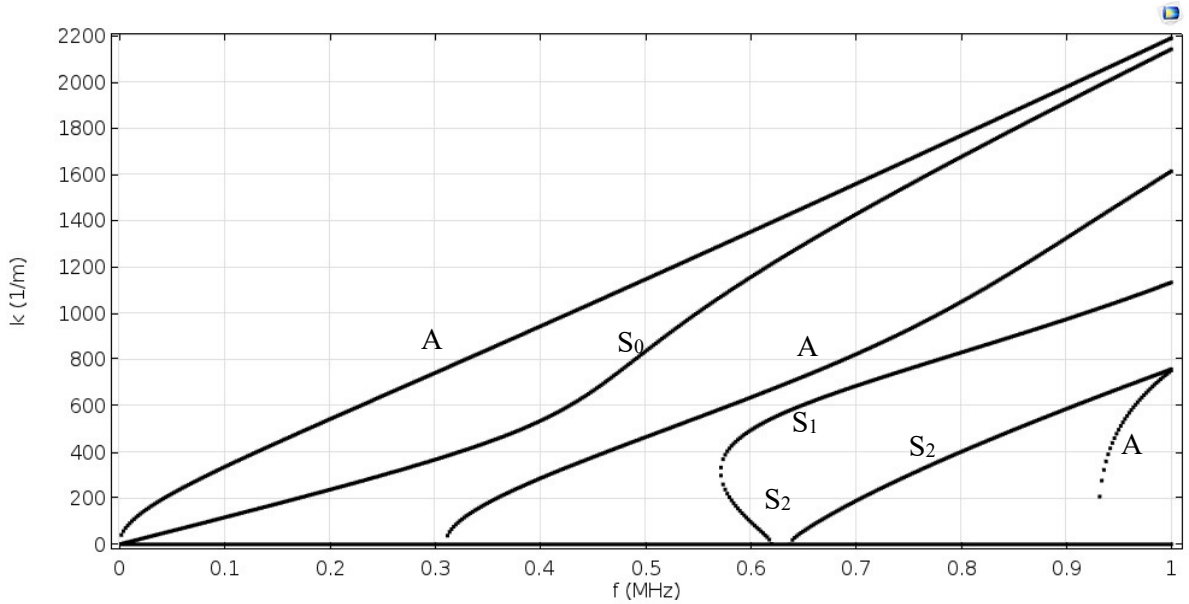

Figure 2. Wavenumbers of the 5 mm thick aluminum plate

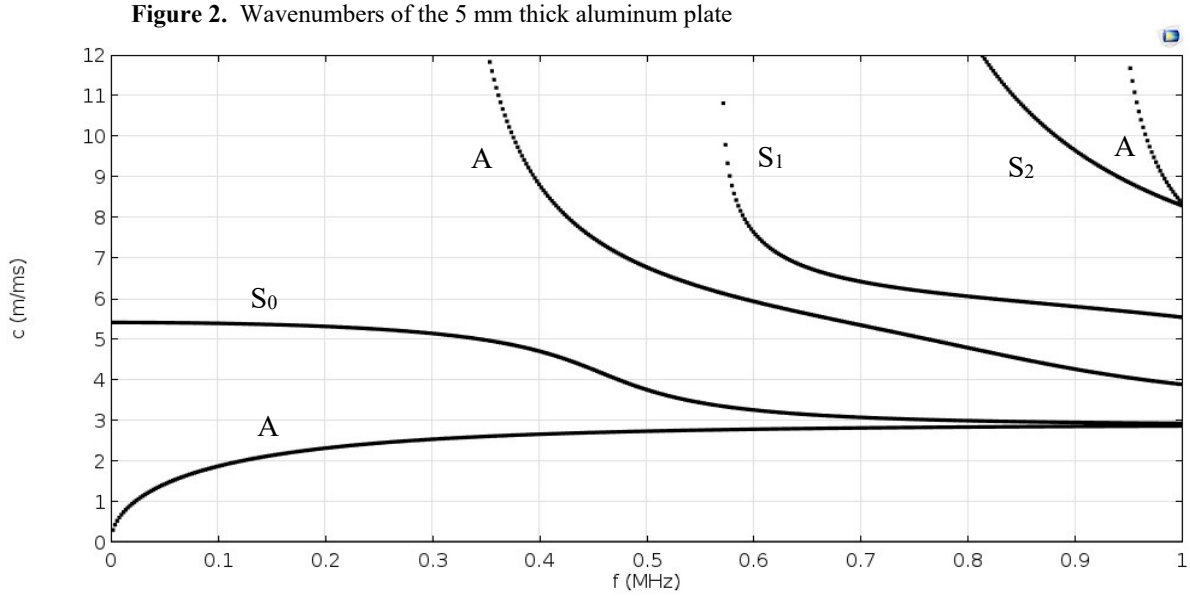

Figure 3. Phase velocities of the 5 mm thick aluminum plate

eigenvalues in decreasing order of the modulus of complex values with negative real parts, followed by those with increasing modulus but with positive real part. For this reason, it is not convenient to follow the modes for wide frequency ranges, over cut-off frequencies when a mode changes its wavenumbers from complex or imaginary to real values. Beyond these frequencies, the modes propagate, theoretically without attenuation. For this reason, we preferred to plot in COMSOL only the positive real values as dots, which are sufficiently close to allow mode identification. This is done by multiplication of the obtained wavenumbers by two logical conditions: "real(lambda)>0" and "abs(imag(lambda)) <1", in which lambda is the implicit name given by COMSOL to the eigenvalues. A mode tracking program is available, but it is written in Matlab, so data communication is required between the two software packages.

As the given example is simple, the mode identification is straightforward (Figure 2) being given in any specialized textbook. One important remark is the separation between the symmetrical modes S_1 and S_2 at about 0.58 MHz in this case. The S_2 mode begins as a so called "backward" mode: the wavenumber decreases with increasing frequency and the group velocity (not shown here) and phase velocity (Figure 3) have opposite signs [9]. The missing part between the two branches of the S_2 mode is in fact purely imaginary and thus not shown. The dispersion curves can also be represented by the phase velocities $c = 2\pi/k$ (Figure 3). One remark is the presence of vertical asymptotes at the cut-off frequencies.

5.2 Energy flux

According to formula (22), the energy flux function along the cross-section can be expressed in COMSOL as the following expression:

$$\Phi(y)=\text{pi}*\text{freq} \\ *\text{imag}((\text{C11}*\text{i}*\text{u1}+\text{C12}*\text{vx})*\text{conj}(\text{u}) \\ +\text{C66}*(\text{ux}+\text{i}*\text{v1})*\text{conj}(\text{v})) \qquad (24)$$

Since $u_1 = ku;\ v_1 = kv$, these shorter expressions were used in defining the strains. One can thus investigate the energy flux for any mode at any frequency. It is only required to extract the eigenvalue index for the selected mode at the chosen frequency, from the wavenumbers plot at the chosen frequency. Then plotting expression (24) on a new line graph window, for the frequency selected from a list and manual eigenvalue selection, provides graphs like the one on Figure 4.

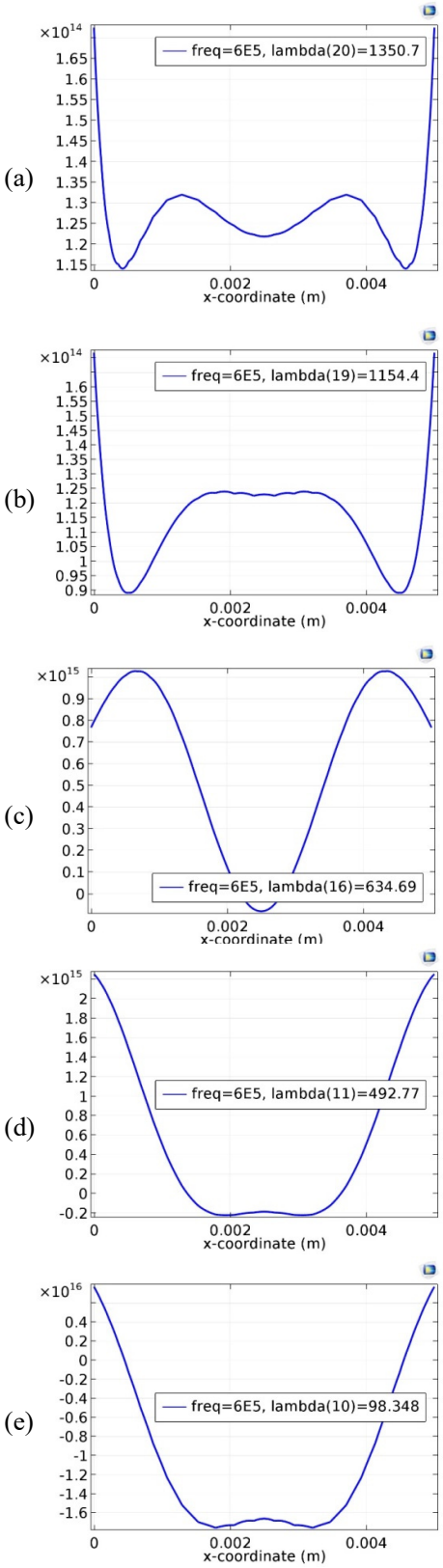

Figure 4. Energy flux distribution along the plate cross-section at 600 kHz: mode A0 (a), S0 (b), A1 (c), S1 (d), S2 (e)

One remark concerns the magnitude of flux distribution range in W/m^2 multiplied by an arbitrary unit (A.U.) of identical magnitude order for all modes. The A.U. comes from the COMSOL modal displacements. Usually the modal displacements are normalized by their Root Mean Square (RMS). As an option, the displacements can be normalized by the function to be integrated is (24) multiplied by the logical conditions mentioned in the previous paragraph. This product eliminates non-propagating modes. The results are shown on Figure 5. The fundamental modes have practically identical fluxes above 475 kHz, resembling to Rayleigh waves.

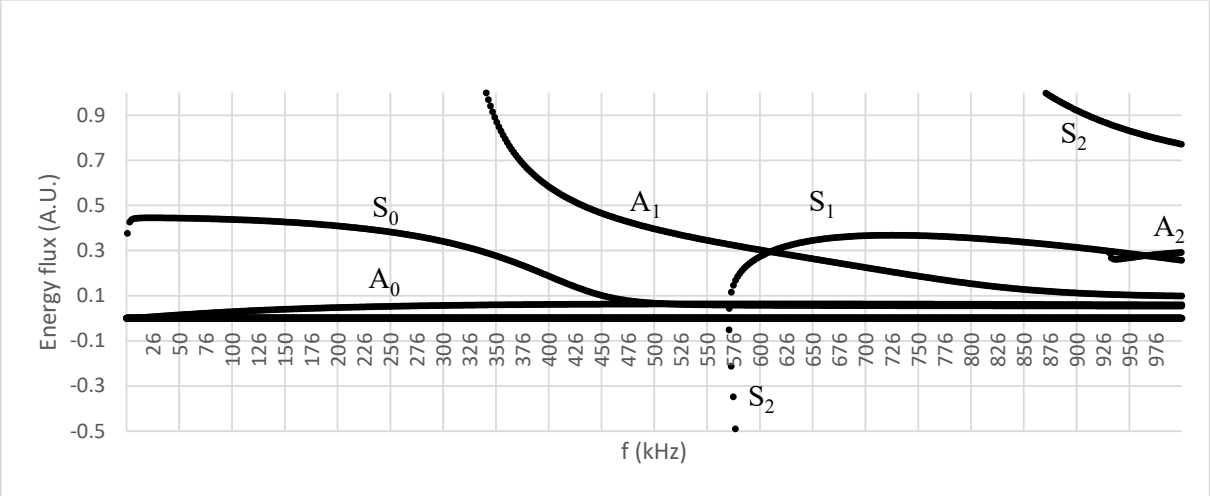

Figure 5. Energy flux averaged on the plate cross-section

their maximum value. In these conditions, the modal displacements are of the order of unity, in which implicitly the unit is 1 m. Consequently, the strains are large and then multiplied by the physical values of elasticity constants. In these conditions the A_0 and S_0 modes have energy fluxes of the order 10^{14}, for A_1 and S_1 of the order 10^{15} and S_2 of the order 10^{16}. These results can be interpreted as required energy flux required to obtain almost the same displacements. Other authors are dividing the computed displacements by the associated averaged energy flux.

Thus, the displacements of the fundamental modes (A_0 and S_0) are considerably larger than those of higher order modes. The fundamental modes have maxima of flux distributions confined to the plate free surfaces at this frequency. The higher order modes exhibit subdomains with local negative values of the flux distribution, but the integral will prove to remain positive, indicating the energy flux propagating in the same sense as the wavenumber which is positive. The S_2 mode has the highest values of $\Phi(y)$ in absolute value, but mostly negative. Indeed, this result confirms the "backward" mode for this mode at this specific frequency.

In order to have an overall image, the integral (22) can be computed by defining in the "Definitions" of "Component 1" of COMSOL, an integration operator along the cross-section line segment. Then in "Results/Derived values/Integral"

Right after the cut-offs, since the phase velocities are asymptotic, so are the fluxes. The "backward" branch of the S_2 mode is also clearly visible with negative energy flux, relative to the positive phase velocity. It can be remarked that S_1 mode between 600-900 kHz requires a similar energy flux as S_0 in the low frequency domain.

These results can be used in interpreting experimental results. Extending the algorithm to multilayered structures is very simple. It suffices to define a line segment for each layer and set the elastic or visco-elastic constants for each layer.

4. CONCLUSIONS

The novelty of this paper is to explicitly present the SAFEM algorithm implemented in COMSOL providing the dispersion curves for guided waves of homogeneous isotropic plates, with particular emphasis on computing the energy flux as function of frequency. This last aspect can be used in interpreting experimental data. The algorithm can be easily extended for multilayered plates (composites), using a more general material model, if the layers are not isotropic.

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Annex A.

The time average over a period for the functions involved in the energy flux, uses the following simple integrals:

$$\begin{aligned}
&\frac{1}{T}\int_t^{t+T}\cos^2(\omega t)\,dt=\frac{1}{2};\\
&\frac{1}{T}\int_t^{t+T}\sin^2(\omega t)\,dt=\frac{1}{2};\\
&\frac{1}{T}\int_t^{t+T}\sin(\omega t)\cos(\omega t)\,dt=0
\end{aligned} \tag{25}$$

Annex B.

Three formulations of the functions involved in the energy flux formula (20). The first two terms in the integral can be expressed as:

$$\begin{aligned}
&S_{xx}^r U^i - S_{xx}^i U^r = \\
&\mathrm{Im}\left[S_{xx}^r U^r + S_{xx}^i U^i + i\left(S_{xx}^r U^i - S_{xx}^i U^r \right) \right] \\
&= \mathrm{Im}\left[\left(S_{xx}^r - iS_{xx}^i \right)\left(U^r + iU^i \right) \right] \\
&= \mathrm{Im}\left[\overline{\left(S_{xx}^r + iS_{xx}^i \right)}\left(U^r + iU^i \right) \right] = \mathrm{Im}\left(\bar{S}_{xx} U \right)
\end{aligned} \tag{26}$$

Another form can be obtained, using the complex conjugate of the displacement field:

$$\begin{aligned}
&S_{xx}^r U^i - S_{xx}^i U^r \\
&= -\mathrm{Im}\left[S_{xx}^r U^r + S_{xx}^i U^i - i\left(S_{xx}^r U^i - S_{xx}^i U^r \right) \right] \\
&= -\mathrm{Im}\left[\left(S_{xx}^r + iS_{xx}^i \right)\left(U^r - iU^i \right) \right] \\
&= -\mathrm{Im}\left[\left(S_{xx}^r + iS_{xx}^i \right)\overline{\left(U^r + iU^i \right)} \right] = -\mathrm{Im}\left(S_{xx}\bar{U} \right)
\end{aligned} \tag{27}$$

A combined form can be obtained from the previous two formulas:

$$S_{xx}^r U^i - S_{xx}^i U^r = \frac{1}{2}\mathrm{Im}\left[\bar{S}_{xx} U - S_{xx}\bar{U} \right] . \tag{28}$$

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